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THEOREM 11. (SCHOUTE'S theorem) *In any triangle, the locus of a point whose pedal triangle has a constant Brocard angle is a circle coaxial with the circumcircle and the Brocard circle.*

The proof is effected by performing the inversion of theorem 9, noting theorem 7, and applying Theorem 10.

COROLLARY. *In particular, the locus of a point whose pedal triangle has the same Brocard angle (in the same sense) as the given triangle is the Brocard circle; and for the same Brocard angle but with the triangle described in the opposite sense, the locus is the Lemoine line.*

For, in the first place, the Brocard circle contains a point O, whose pedal triangle is similar to the given triangle; and further, the points of this circle and of the Lemoine line are mutually inverse with regard to the circumcircle.

All previous proofs by elementary methods, even of the corollary, have made use of a very awkward method of vertical projections from one plane to another (cf. Gallatly, l. c., or Emmerich, Die Brocard'sche Gebilde, §§ 60, 61). It will be noted that most of the steps of the present proof may fairly be regarded as general theorems, very little special proof being required for the theorem itself.

A SIMPLE RELATION BETWEEN ELEMENTARY NUMBER-THEORY AND ELEMENTARY PROJECTIVE GEOMETRY.

By AUBREY J. KEMPNER, Urbana, Ill.

In *L'Enseignement Mathématique*, 1916, p. 332, Mr. A. Reymond has published a note in which, with slight modifications, the following simple graphical method of determining whether a given integer is a prime number, is explained.

Write in a vertical and in a horizontal row the numbers 0, 1, 2, 3, 4, $\cdots$, as far as one likes, and fill out Fig. 1 in the way indicated. Evidently every number in our first vertical row, for example 6, will have for factors just the numbers which are in the same horizontal row, in this case 1, 2, 3, 6. Here the number 1 and the number 6 are both considered factors of 6. The number 7 has only the

0	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	2								
3	1		3							
4	1	2		4						
5	1				5					
6	1	2	3			6				
7	1						7			
8	1	2		4				8		
9	1		3						9	
10	1	2			5					10

FIG. 1.

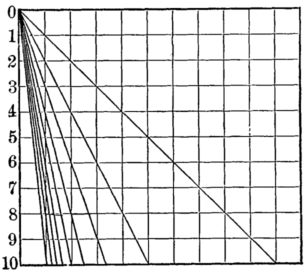

FIG. 2.

two factors 1 and 7 and is therefore prime. In general, a number is prime when, and only when, in the same horizontal row with it, there are exactly two numbers. (The number 0, which contains every integer $\neq 0$ as a factor, and the number 1, are exceptional, as can be read off from Fig. 1.)

The method by which Fig. 2 is derived from Fig. 1 is obvious and requires no explanation. If we call the points in Fig. 2, in which exactly three straight lines intersect, knot-points, for brevity, then we see that a number is a prime number when and only when the horizontal line belonging to it in the figure has exactly two knot-points (excepting the special numbers 0, 1). In general, the number of distinct divisors of n (including 1 and n among the divisors) is exactly equal to the number of knot-points lying on the corresponding horizontal line.

For practical purposes, the construction is without value (and this quite apart from the fact that its application to any but small numbers becomes impossible on account of the complexity of the figure), because we have no means of deciding, in an empirical drawing, whether three lines meet in a point, or in three points forming the vertices of a small triangle. Also, as Mr. Reymond points out, it seems unlikely that results of theoretical interest may be obtained by a closer study of the construction (which is essentially a graphical interpretation of the method known under the name "Sieve of Eratosthenes." See any book on elementary number-theory). In spite of these objections, the construction is of interest as bringing out clearly the fact that the problem of deciding whether a given integer is a prime number, may be graphically solved by means of ruler and compasses. When one remembers, however, that only rational operations are required in order to determine the prime character of a number n (trial-divisions by integers less than n; more precisely trial-divisions by all prime numbers equal to or less than the square root of n) and even to determine all of its factors, it is clear that a linear construction with straight-edge[1] alone should be possible.

For justification of this statement, in case it should not be evident, see, for example, Weber-Wellstein, referred to below, pp. 210, 211.

To obtain such a construction, we shall only have to construct a figure perspective to Fig. 2. For readers not familiar with the idea of a projective scale, an explanation and references are given.

I. *Projective Scales.*[2] Consider an ordinary metric scale with a set of equidistant points representing for example the positive integers and 0. We wish

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[1] Ruler without scale and without marks of any kind; an instrument used solely for drawing straight lines.

[2] See for example: YOUNG, "The theory of sets of points," 1906, p. 9, or WEBER-WELLSTEIN, "Encyklopädie der Elementar-Mathematik," Vol. II, 2d ed., 1907, p. 198. STEINER ("Die geometrischen Constructionen ausgeführt mittelst der geraden Linie und eines festen Kreises," Berlin, 1833. See "Gesammelte Werke," Vol. I, p. 499, or "Ostwald's Klassiker, Nr. 60," Chapter 3) showed that it is possible to carry out all constructions which may be made by use of straight-edge and compasses (equivalent to graded ruler and compass, provided a fixed unit is given) by means of a straight-edge alone, provided one fixed circle and its center are given. D. CAUER (*Mathematische Annalen*, 1912–13, Vol. 73, p. 90) proved that the center of the circle cannot be dispensed with.

to construct an equivalent projective scale. Introducing on our ordinary scale the point at infinity (∞) as representing the ∞ of real numbers (where no distinction is made between $+\infty$ and $-\infty$, the assumption being made in projective geometry that each straight line contains *one* point at infinity), we may interpret our metric scale in the following manner:
Forming the cross-ratio

$$(a_1, a_2, a_3, \infty) = \frac{a_1 - a_2}{a_3 - a_2} : \frac{a_1 - \infty}{a_3 - \infty},$$

we find

$$(a_1 a_2 a_3 \infty) = -1,$$

when a_1, a_2, a_3 are any three equidistant points, that is, when $a_3 - a_2 = a_2 - a_1$, since $(a_1 - \infty)/(a_3 - \infty)$ must be counted equal to unity, as one sees by a simple limiting process. Hence the four points a_1, a_2, a_3, ∞, taken in this order, are harmonic points. We make for the construction of a projective scale the following assumptions. Instead of the point at infinity on our metric scale, we choose on our line an arbitrary point u, which shall play on our projective scale the rôle of the point at infinity on our ordinary scale; we accordingly agree to consider three points a_1, a_2, a_3 as equidistant, that is, $a_3 - a_2 = a_2 - a_1$, when the four points a_1, a_2, a_3, u are harmonic points. By repeatedly using the classical construction for the determination of the fourth harmonic point when three points are given we see the truth of the following theorem:

When we are given on a line three points, a_1, a_2, u, which shall represent, respectively, 0, 1, ∞, then it is possible to construct as far as one likes by means of straight-edge alone[1] a projective scale containing the points representing 0, 1, 2, 3, 4, 5, $\cdots$.[2] The construction of the scale is unique, when 0, 1, ∞ are arbitrarily given; this fact may also be expressed by saying that a projective scale is completely determined by three of its points.

It is also possible to obtain, by straight-edge construction, a projective scale containing as many "rational" points as one likes, but we shall need only the points corresponding to integers. Likewise, it is beyond our purposes to introduce the "negative" numbers of the projective scale. For all this, and for the important question as to what assumptions must be made to ensure continuity in the projective plane, see either Young or Weber-Wellstein, referred to above.

For measurements in projective geometry, these projective scales usually take the place of ordinary scales in metric geometry. The ordinary metric scale is obviously a very special projective scale, obtained by assuming u at infinity and taking the length $a_1 \cdots a_2$ for the unit of length. Note that if 0, 1, 2, 3, 4, $\cdots$, form a projective scale, then also 0, 2, 4, 6, 8, $\cdots$ form a projective scale, and in general $0, k, 2k, 3k, 4k, \cdots$ form a similar scale. (For our purposes k may be assumed a positive integer.)

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[1] For the construction of the ordinary metric scale, where u is actually the point at infinity on the line, compasses cannot be avoided. Compare, however, the reference to Steiner given above.

[2] We shall generally call these points simply 0, 1, 2, 3, $\cdots$, or $0'$, $1'$, $2'$, $3'$, $\cdots$, or $0''$, $1''$, $2''$, $3''$, $\cdots$, while the point u will be denoted by ∞, or ∞', or ∞''.

II. We make now the following construction (compare Fig. 3):

In any triangle ABC, construct on BC a projective scale, taking $B = 0$, $C = \infty$, and an arbitrary point between B and C for 1. Similarly construct on AB a projective scale, taking $B = 0$, $A = \infty$, and an arbitrary point between B and A for 1. The points on the scale BC may be called $0, 1, 2, 3, 4, \cdots$, the points on the scale AB, $0', 1', 2', 3', 4', \cdots$. Assume the rays $A1, A2, A3, A4, \cdots$ drawn, and also the rays $C1', C2', C3', C4', \cdots$. The point of intersection of two rays An, Cm' may be denoted by (n, m'). Then it follows from fundamental theorems of projective geometry that the points $(0, 0') = B$, $(1, 1')$, $(2, 2')$, $(3, 3')$, $(4, 4')$, $\cdots$ all lie in a straight line l_1. Taking into account the remark at the end of I, we see also that $(0, 0')$, $(2, 1')$, $(4, 2')$, $(6, 3')$, $(8, 4')$, $\cdots$ all lie in a straight line l_2, and in general $(0, 0')$, $(k, 1')$, $(2k, 2')$, $(3k, 3')$, $(4k, 4')$, $\cdots$, (k any positive integer) all lie in a straight line l_k. These lines $l_1, l_2, l_3, \cdots l_k \cdots$ all pass through B; denoting the intersection of l_k and AC by k'', it is now easy to see that, provided we call $A = 0''$[1] and call $C = \infty''$, the points $0''$, $1''$, $2''$, $3''$, $4''$, $\cdots$ on AC also form a projective scale.

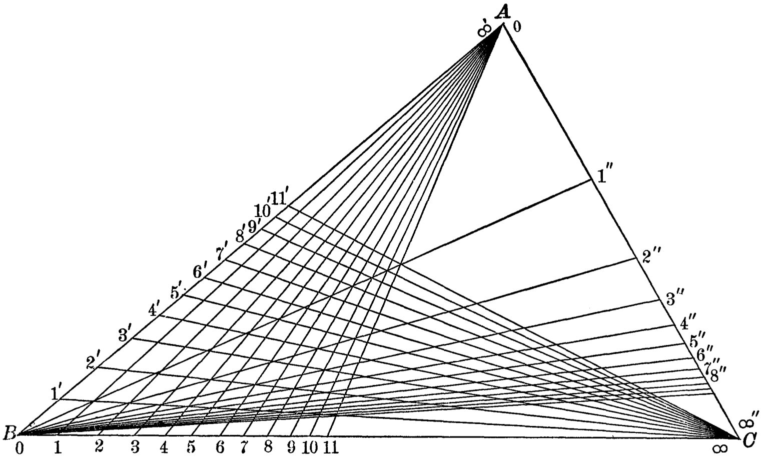

Fig. 3.

III. From what has been said, we may derive the following result (see Fig. 3):

Assume any triangle ABC, choose on BC any point 1, and construct the projective scale $0, 1, 2, 3, \cdots$ on BC, as far as one likes, taking $B = 0$ and $C = \infty$; similarly, construct the projective scale $0', 1', 2', 3', \cdots$ on BA, choosing any point $1'$ on AB, and taking $B = 0'$, $A = \infty'$. Draw the rays $A1, A2, \cdots An \cdots$ and $C1', C2', \cdots Cm' \cdots$. Call the intersection of An and Cm' (n, m'). Draw the ray $B(1, 1')$. This ray intersects AC in a point which we call $1''$. Construct the projective scale $0''$, $1''$, $2''$, $3''$, $4''$, $\cdots$ on AC, taking $A = 0''$, $C = \infty''$, and draw the rays $B2''$, $B3''$, $B4''$, $\cdots$. We thus have in our triangle a set of rays from each vertex.[1] Mark in the figure all points in which three rays meet. These points we shall call, for convenience, knot-points. By comparing Fig. 3 with Fig. 1 and Fig. 2, we find:

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[1] This does not contradict our assumption, made above, that A represent ∞', since we were then considering A as a point on AB, while we are now considering A as a point on AC. The two determinations are independent of each other.

The number r of knot-points on An, $n = 2, 3, 4, 5, \cdots$, gives the number of distinct divisors of n, counting both 1 and n as divisors. Therefore n is prime, when and only when there are two knot-points on An. When $r = 3$, n is the square of a prime number, and, in a similar way, some other elementary results (for example the determination of common prime factors of given integers) may be read off the figure.—Readers familiar with projective geometry will recognize immediately how Fig. 3 may be derived from Fig. 2.

Other projective geometric constructions for determining the prime-character of a given number are easily devised, but the construction given above has the advantage of involving only straight lines.

When one ignores simple relations like the one treated in this note, between projective geometry and number theory, which in reality are only analogies without deeper import, the problem of finding connections between projective geometry and number theory seems to be, according to an interesting remark by Klein (Elliptische Modulfunktionen, Vol. I, p. 242, footnote), a task of great difficulty. (See also Weber-Wellstein, II, 1907, pp. 211–212, footnote.)

CONCERNING PREFERENTIAL VOTING.

By L. L. DINES, University of Saskatchewan.

In a recent number of the MONTHLY, Professor W. V. Lovitt has given an interesting discussion of a problem which may be stated as follows:

In a certain election, each voter expresses his first, second, and third choice. The three candidates A, B, and C receive respectively A_i, B_i, and C_i votes for ith choice. It is required to determine what weights may be assigned to the votes for first, second, and third choice in order that A may win.

While the purely algebraic discussion given in the present note is less vivid than the geometric discussion of Professor Lovitt, it may be of interest as it leads more directly to the necessary conditions for the existence of a solution, and to explicit ranges of possible values for the weights in terms of the data A_i, B_i, and C_i. Furthermore, the algebraic method can be extended to a treatment of the more general problem in which there are n candidates, and each voter

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[1] The $\triangle ABC$ now represents, for purposes of projective measurement, the whole first quadrant in an ordinary system of coördinates. It is interesting to see how, by taking on the sides of our triangle (extending each indefinitely in both directions) the "negative" and the "rational" points into account, the whole ordinary plane is covered by a "projective" plane.